

Infinite planar string: cusps, braids and soliton excitations

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keywords: strings, braids, anyon models.

MSC codes: 74K05, 81T30

Abstract

We investigate infinite strings in $(2 + 1)D$ space-time, which may be considered as excitations of straight lines on the spatial plane. We also propose the hamiltonian description of such objects that differs from the standard hamiltonian description of the string. The hamiltonian variables are separated into two independent groups: the "internal" and "external" variables. The first ones are invariant under space-time transformations and are connected with the second form of the world-sheet. The "external" variables define the embedding of the world-sheet into space-time. The constructed phase space is nontrivial because the finite number of constraints entangles the variables from these groups. First group of the variables constitute the coefficients for the pair of first-order spectral problems; the solution of these problems is necessary for the reconstruction of the string world-sheet. We consider the excitations, which correspond to "N- soliton" solution of the spectral problem, and demonstrate that the reconstructed string has cuspidal points. World lines of such points form braids of various topologies.

1 Introduction

The interest for the dynamical systems on a plane was aroused in connection with the idea of topological quantum calculations [1]. The planar string is the example of such system; we suppose that the excitations of the string may be considered as some quasiparticles with anyon statistics. Recently, the non-standard point of view on elementary particles as the defects of the string condensed matter ("string-net condensation") was suggested [2]. Indirectly, our approach corresponds to these ideas.

The cases of both closed and open string in arbitrary space-time dimensions were well-investigated in the literature (see, for example, [3, 4]). One of the frequently discussed objects here is the first form **I** of the world-sheet – as opposed to the second form **II**. In this work we investigate the excitations of an infinite string on a spatial plane in terms of second form **II** and discuss a possible interpretation of such excitations as some anyon-type quasiparticles. The complete classical theory in terms of non-standard hamiltonian variables will be constructed here. The suggested scheme is based on the geometrical approach [5] that is a generalization of a standard geometrical approach [6] in a string theory. Certain ideas of this article can be founded in the work [7] and [8].

The considered strings can be viewed as embedded in four-dimensional space-time; that is why the main points of the approach [7] developed for $4D$ relativistic string should be outlined. We start with the Polyakov action

$$S = -\frac{\gamma}{2} \int \sqrt{h} h^{ij} \partial_i X_\mu \partial_j X^\nu d\xi^0 d\xi^1, \quad (1)$$

where the following notations are introduced: h^{ij} – metric tensor field in two-dimensional space-time (ξ^0, ξ^1) , $h = \det(h^{ij})$ and $X_\mu(\xi^0, \xi^1)$ – scalar fields in two-dimensional space-time with isotopic index μ . The isotopic space is Minkowski space-time $E_{1,3}$ with the metric tensor $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. The standard procedure [4] leads to the equalities

$$\partial_+ \partial_- X_\mu = 0, \quad (\partial_\pm X_\mu)^2 = 0 \quad (\partial_\pm = \partial/\partial\xi_\pm, \xi_\pm = \xi^1 \pm \xi^0),$$

and so the objects of our consideration will be time-like world-sheets with orthonormal parametrization. We consider the infinite strings, that's why we must impose asymptotic conditions on a curve $X_\mu = X_\mu(\xi^1)$ for any value of evolution parameter ξ^0 . The demanded conditions will be formulated later; in the first place we'll investigate the local structure of the world-sheet.

Let us define the pair of light-like vectors in space $E_{1,3}$:

$$\mathbf{e}_\pm(\xi_\pm) = \pm \frac{1}{\varkappa} \partial_\pm \mathbf{X}(\xi_\pm), \quad (2)$$

where \varkappa is an arbitrary positive constant. It is clear that we can construct the pair of orthonormal bases $\mathbf{e}_{\nu\pm}(\xi_\pm)$ that are connected with the introduced vectors \mathbf{e}_\pm by the equalities $\mathbf{e}_\pm = (\mathbf{e}_{0\pm} \mp \mathbf{e}_{3\pm})/2$. These bases allow us to define the vector-matrices $\hat{\mathbf{E}}_\pm$:

$$\hat{\mathbf{E}}_\pm = \mathbf{e}_{0\pm} \mathbf{1}_2 - \sum_{i=1}^3 \mathbf{e}_{i\pm} \boldsymbol{\sigma}_i. \quad (3)$$

The stationary groups for the vectors \mathbf{e}_\pm will be three - parameter groups. Consequently, the definition of the bases $\mathbf{e}_{\nu\pm}(\xi_\pm)$ has three - parameter arbitrariness in each point (ξ^0, ξ^1) . We will keep this fact in mind and eliminate this ambiguity in the next section. The important object for our subsequent considerations is the $SL(2, C)$ - valued field $K(\xi^0, \xi^1)$. This field is defined as follows:

$$\hat{\mathbf{E}}_+ = K \hat{\mathbf{E}}_- K^+ . \quad (4)$$

To fulfill the reduction into $D = 1+2$ space-time we must require the matrix K to be real, so that $K \in SL(2, R)$. This requirement means that

$$\mathbf{e}_{2+}(\xi_+) = \mathbf{e}_{2-}(\xi_-) = \mathbf{b}_2 ,$$

where \mathbf{b}_2 is a constant spatial vector so that the reduced space-time $E_{1,2} \perp \mathbf{b}_2$.

In accordance with the definition of the vector-matrices $\hat{\mathbf{E}}_\pm$, these matrices satisfy the equalities $\partial_\pm \hat{\mathbf{E}}_\mp = 0$. As the consequence, the matrix field $K(\xi^0, \xi^1)$ satisfies to (special) WZWN - equation

$$\partial_+ (K^{-1} \partial_- K) = 0 . \quad (5)$$

Let us define the real functions $\varphi(\xi^0, \xi^1)$ and $\alpha_\pm(\xi^0, \xi^1)$ by means of Gauss decomposition for the matrix $K(\xi^0, \xi^1)$:

$$K = \begin{pmatrix} 1 & 0 \\ -\alpha_+ & 1 \end{pmatrix} \begin{pmatrix} \exp(-\varphi/2) & 0 \\ 0 & \exp(\varphi/2) \end{pmatrix} \begin{pmatrix} 1 & \alpha_- \\ 0 & 1 \end{pmatrix} . \quad (6)$$

In general, these functions are singular because the decomposition (6) is not defined for the points where the principal minor $K_{11} = 0$. Let us introduce the regular functions $\rho_\pm = (\partial_\pm \alpha_\mp) \exp(-\varphi)$. The consequence of the equality (5) will be the following PDE - system:

$$\partial_+ \partial_- \varphi = 2\rho_+ \rho_- \exp \varphi, \quad (7a)$$

$$\partial_\pm \rho_\mp = 0, \quad (7b)$$

$$\partial_\pm \alpha_\mp = \rho_\pm \exp \varphi. \quad (7c)$$

The introduction of the function φ and the functions ρ_\pm is justified by the following formulae for the first (**I**) and the second (**II**) forms of the world-sheet:

$$\mathbf{I} = -\frac{\varkappa^2}{2} e^{-\varphi} d\xi_+ d\xi_-, \quad \mathbf{II} = \varkappa [\rho_+ d\xi_+^2 - \rho_- d\xi_-^2] .$$

The standard method of geometrical description of a string [6] uses the equations (7a) and (7b) which are deduced from the Gauss and Peterson-Kodazzi equations. In a standard approach the inequalities $\rho_{\mp} > 0$ are fulfilled. In this case the conformal transformations

$$\xi_{\pm} \longrightarrow \tilde{\xi}_{\pm} = A_{\pm}(\xi_{\pm}), \quad A' \neq 0, \quad (8)$$

allow to reduce the equation (7a) to Liouville equation; the function $\varphi(\xi^0, \xi^1)$ will be the only dynamical variable here. In our approach there are no restrictions for the real functions ρ_{\mp} which will be the dynamical variables too. For example, the identity $\rho \equiv 0$ should take place on any interval $[a, b]$. We must emphasize that in this case there are no transformations (8) that reduce the equation (7a) to Liouville equation globally.

Let us point out the principal differences between Liouville equation and system (7). These differences are:

1. The system (7) has a "good" translate-invariant solution $\varphi \equiv \text{const}$, $\rho_{\pm} \equiv 0$ (classical vacuum). It should be reminded that the corresponding solution for Liouville equation is unsatisfying because $\varphi \equiv -\infty$. This situation for Liouville theory leads to some problems, in a quantum theory for example.
2. The group \mathcal{G} of the system invariancy is essentially wider then the group (8). Indeed, let the functions $\varphi(\xi_+, \xi_-)$, $\rho_{\pm}(\xi_{\pm})$ and $\alpha_{\pm}(\xi_+, \xi_-)$ be solutions of the system (7). Then the transformation

$$(\varphi, \rho_{\pm}, \alpha_{\pm}) \longrightarrow (\tilde{\varphi}, \tilde{\rho}_{\pm}, \tilde{\alpha}_{\pm}), \quad (9)$$

gives the new solution of the system (7) if

$$\begin{aligned} \tilde{\varphi}(\xi_+, \xi_-) &= \varphi(A_+(\xi_+), A_-(\xi_-)) + f_+(\xi_+) + f_-(\xi_-), \\ \tilde{\rho}_{\pm}(\xi_{\pm}) &= \rho(A_{\pm}(\xi_{\pm})) A'_{\pm}(\xi_{\pm}) \exp(-f_{\pm}(\xi_{\pm})), \\ \tilde{\alpha}_{\pm}(\xi_+, \xi_-) &= \alpha_{\pm}(A_+(\xi_+), A_-(\xi_-)) \exp(f_{\pm}(\xi_{\pm})) + g_{\pm}(\xi_{\pm}). \end{aligned}$$

for arbitrary real functions $f_{\pm}(\xi)$, $g_{\pm}(\xi)$ and such real functions $A_{\pm}(\xi)$ where the conditions $A'_- A'_+ \neq 0$ take place.

3. The system (7) has a different hamiltonian structure [9].

From the geometrical point of view, two kinds of the transformations (9) exist. First kind ones correspond to the conformal reparametrizations of the same world-sheet. The equalities

$$f_{\pm}(\xi) = -\ln A'_{\pm}(\xi) \quad (10)$$

extract these transformations from the group \mathcal{G} . Second kind are the rest transformations which connect the different world-sheets.

In accordance with point one, we will use the following asymptotical conditions in our theory:

$$\lim_{\xi^1 \rightarrow \pm\infty} \varphi(\xi^1) = C_{\pm}, \quad (11)$$

$$\lim_{\xi \rightarrow \pm\infty} \rho(\xi) = 0. \quad (12)$$

The geometrical meaning of these conditions is as follows: we consider the world-sheets that are planes on spatial infinities. In spite of simplicity and naturalness of such asymptotical behavior for a world-sheet, these conditions are unusual in a string theory. So, the integral curvature of a world-sheet in our approach is finite:

$$\mathcal{K} = \int k dS < \infty. \quad (13)$$

The quantity $k = \det \mathbf{II} / \det \mathbf{I}$ is a Gauss curvature here. Note that the integral curvature will be \mathcal{G} - an invariant value. For subsequent purposes we must reinforce the supposition (12). We suppose that the functions $\rho_{\pm}(\xi)$ are the functions from Shwarz space.

Let us pay attention to arbitrary constant \varkappa which seemed to be an unnecessary value in the definition (2) before. Taking into account the formula for the first world-sheet form (\mathbf{I}), we justify the introduction of the constant \varkappa by demanding $C_- = 0$ in our subsequent considerations. Thus the constant \varkappa parameterize the "classical vacua" set for system (7). Moreover we demand that

$$\lim_{\xi^1 \rightarrow -\infty} K(\xi^0, \xi^1) = 1_2. \quad (14)$$

That's why we restrict the transformations (9) by the condition

$$\lim_{\xi^1 \rightarrow -\infty} (f_+(\xi^1 + \xi^0) + f_-(\xi^1 - \xi^0)) = 0 \quad (15)$$

and

$$\lim_{\xi^1 \rightarrow -\infty} g_{\pm}(\xi^1 \pm \xi^0) = 0 \quad (16)$$

for the functions f_{\pm} and g_{\pm} .

2 Factorization of the world-sheet set.

Let the vectors \mathbf{b}_{μ} be constant vectors so that $\mathbf{b}_{\mu}\mathbf{b}_{\nu} = g_{\mu\nu}$. Let the vector - matrix $\hat{\mathbf{E}}_0 = \mathbf{b}_0\mathbf{1}_2 - \sum_{i=1}^3 \mathbf{b}_i\sigma_i$ correspond to the basis \mathbf{b}_{μ} . It is clear that

$$\hat{\mathbf{E}}_{\pm}(\xi_{\pm}) = T_{\pm}(\xi_{\pm})\hat{\mathbf{E}}_0T_{\pm}^{\top}(\xi_{\pm}), \quad (17)$$

where $T_{\pm}(\xi) \in SL(2, R)$. In accordance with formula (4) the equality

$$K(\xi^0, \xi^1) \equiv T_+(\xi_+)T_-^{-1}(\xi_-). \quad (18)$$

takes place. The following proposition can be deduced directly from the definitions of the matrices T_{\pm} and K :

Proposition 1. *The matrices T_{\pm} are the solutions for the linear problems*

$$T'_{\pm}(\xi) + Q_{\pm}(\xi)T_{\pm}(\xi) = 0, \quad (19)$$

where

$$Q_-(\xi^0, \xi^1) = K^{-1}\partial_-K, \quad Q_+(\xi^0, \xi^1) = -(\partial_+K)K^{-1}. \quad (20)$$

The global Lorenz transformations in our theory are the transformations

$$\hat{\mathbf{E}}_0 \longrightarrow \tilde{\mathbf{E}}_0 = B\hat{\mathbf{E}}_0B^{\top}, \quad (21)$$

where the constant matrix $B \in SL(2, R)$. It is clear that these transformations correspond to the arbitrariness for the matrix - solution of the systems (19):

$$T_{\pm} \longrightarrow \tilde{T}_{\pm} = T_{\pm}B_{\pm}, \quad B_{\pm} \in SL(2, R),$$

where $B_+ = B_- = B$ in accordance with the formulae (14) and (18). Thus the coefficients of the problems (19) are local functions of the introduced variables φ , ρ_{\pm} and α_{\pm} . These coefficients are relativistic invariants. For example, the equalities

$$Q_{12+} = -\rho_+, \quad Q_{21-} = -\rho_-, \quad (22)$$

will be important for our subsequent considerations. Our investigations are devoted to the string excitations that should be appropriate candidates for some anyon-type quasiparticles. That's why we restrict the transformations (21) by demanding $B \in SO(2)$ in the considered model. So, we can write that

$$T_{\pm}(\xi) = T_{0\pm}(\xi)U(\beta), \quad (23)$$

where

$$T_{0\pm}(\xi) \in SL(2, R), \quad \lim_{\xi \rightarrow -\infty} T_{0\pm}(\xi) = 1_2, \quad U(\beta) \in SO(2).$$

Let \mathcal{G}_0 be subgroup of the group \mathcal{G} so that $A_{\pm}(\xi) \equiv \xi$ for all transformations (9). Then the following proposition is true:

Proposition 2. *If the group \mathcal{G}_0 acts on a solution $\{\varphi, \rho_{\pm}, \alpha_{\pm}\}$ of the system (7), the matrices T_{\pm} are transformed as follows:*

$$T_{\pm} \longrightarrow \tilde{T}_{\pm} = G_{\pm}^{-1} T_{\pm}, \quad (24)$$

where

$$G_{+} = \begin{pmatrix} e^{f_{+}/2} & 0 \\ g_{+}e^{-f_{+}/2} & e^{-f_{+}/2} \end{pmatrix}, \quad G_{-} = \begin{pmatrix} e^{-f_{-}/2} & g_{-}e^{-f_{-}/2} \\ 0 & e^{f_{-}/2} \end{pmatrix}.$$

Proof is direct consequence of the formulae (6), (18) and explicit form for the transformations (9). Let $\mathcal{G}_0[T]$ denote the orbit of the group \mathcal{G}_0 for matrix $T = T_{\pm}(\xi)$. Then the following corollaries are fulfilled.

Corollary 1. *The unique matrix $\mathcal{U} \in \mathcal{G}_0[T]$ exists so that $\mathcal{U} \in SO(2)$.*

Indeed, let us consider the Iwasawa decomposition for matrices $T_{\pm}(\xi)$ so that $T_{\pm} = \mathcal{E}_{\pm} N_{\pm} U_{\pm}$ where the matrices \mathcal{E}_{\pm} are diagonal matrices with positive elements, N_{+} (N_{-}) – lower (upper) triangular matrix and $U_{\pm} \in SO(2)$. Because of the unique existence of such decomposition for any matrix $T \in SL(2, R)$ and rule (24) for the matrix T_{\pm} transformation the proof of the Corollary 1 is clear.

Corollary 2. *The matrices $\mathcal{U}_{\pm}(\xi)$ satisfy the linear problems*

$$\mathcal{U}'_{\pm}(\xi) + Q_{\pm}(\xi)\mathcal{U}_{\pm}(\xi) = 0, \quad (25)$$

where matrices Q_{\pm} are as follows: $Q_{\pm} = -\rho_{\pm}\sigma_{\pm} + \rho_{\pm}\sigma_{\mp}$.

We use same characters for matrices Q (similar for coefficients ρ) both in the spectral problem (19) and in the spectral problem (25); we hope that these notations don't lead to any ambiguities.

The following step is the reconstruction of the world-sheet through the matrix elements $t_{ij\pm}$ of the matrices T_{\pm} . Taking into account the formula (17) and the definition of the matrices $\hat{\mathbf{E}}_{\pm}(\xi_{\pm})$, we obtain the following equalities:

$$\pm \partial_{\pm} X(\xi_{\pm}) = \frac{\varkappa}{2} \left[(t_{i1\pm}^2 + t_{i2\pm}^2) \mathbf{b}_0 - 2(t_{i1\pm} t_{i2\pm}) \mathbf{b}_1 - (t_{i1\pm}^2 - t_{i2\pm}^2) \mathbf{b}_3 \right], \quad (26)$$

where index i corresponds to the sign \pm according to the rule $i = \frac{3\mp 1}{2}$.

The group \mathcal{G}_0 can be decomposed into two kinds of the special transformations:

$$\begin{aligned} (A) \quad & \alpha_{\pm} \rightarrow \alpha_{\pm} + g_{\pm}, \\ (B) \quad & \varphi \rightarrow \varphi + f_{+} + f_{-}, \quad \rho_{\pm} \rightarrow \rho_{\pm} e^{-f_{\pm}}, \quad \alpha_{\pm} \rightarrow \alpha_{\pm} e^{f_{\pm}}. \end{aligned}$$

The following proposition is true.

Proposition 3. *The transformation (A) does not change the world-sheet; the transformation (B) transforms the world-sheet to the other one so that*

$$\mathbf{I} \longrightarrow \tilde{\mathbf{I}} = e^{-f} \mathbf{I}, \quad f = f_+(\xi_+) + f_-(\xi_-), \quad (27)$$

and the second form \mathbf{II} :

$$\mathbf{II} \longrightarrow \tilde{\mathbf{II}} = \varkappa[\rho_+ e^{-f_+} d\xi_+^2 - \rho_- e^{-f_-} d\xi_-^2] \quad (28)$$

The proof follows from the explicit formulae (26) for tangent vectors $\partial_\pm X(\xi_\pm)$, explicit formulae for the forms \mathbf{I} and \mathbf{II} and rules (24) for transformations of the matrix elements $t_{ij\pm}$. As a comment we note that the existence of the transformations (A), which do not change the world-sheet, is the consequence of the arbitrariness in the definition of matrices $\hat{\mathbf{E}}_\pm(\xi_\pm)$.

Let \mathcal{X} be the set of a world-sheets introduced in the beginning of the paper. The object of our subsequent investigations is factor-set \mathcal{X}/\mathcal{G} only. Taking into account Corollary 1 we can choose the representatives in every coset so that $T_\pm = \mathcal{U}_\pm \in SO(2)$. In accordance with the formula (23) we have

$$\mathcal{U}_\pm(\xi) = \mathcal{U}_{0\pm}(\xi) U(\beta), \quad \lim_{\xi \rightarrow -\infty} \mathcal{U}_{0\pm}(\xi) = 1_2, \quad (29)$$

where the matrix $\mathcal{U}_{0\pm}(\xi) \in SO(2)$. Because of the boundary conditions on the matrices $\mathcal{U}_{0\pm}$ the one-to-one correspondence $\rho_\pm(\xi) \leftrightarrow \mathcal{U}_{0\pm}(\xi)$ exists for each sign \pm . Let us introduce the quantities $I_\pm(\xi) = \int_{-\infty}^{\xi} \rho_\pm(\eta) d\eta$. It is easy to see that

$$\mathcal{U}_{0\pm}(\xi) = \begin{pmatrix} \cos I_\pm(\xi) & \pm \sin I_\pm(\xi) \\ \mp \sin I_\pm(\xi) & \cos I_\pm(\xi) \end{pmatrix}. \quad (30)$$

In the context of the factorization procedure, as defined above, we are able to write the principal minor K_{11} of the matrix $K(\xi^0, \xi^1)$ as the function of the quantities ρ_\pm . To do it we must extract the element K_{11} from the formula (6). The result is as follows:

$$\exp[-\varphi(\xi^0, \xi^1)] = \cos^2[I_+(\xi_+) + I_-(\xi_-)]. \quad (31)$$

This equality can be considered as the geometrical gauge condition for our theory. It must be emphasized that the arbitrariness (8) is not fixed anywhere.

The formulae for tangent vectors $\partial_\pm \mathbf{X}(\xi_\pm)$ for chosen representatives is as follows:

$$\pm \partial_\pm \mathbf{X}(\xi_\pm) = \frac{\varkappa}{2} [\mathbf{b}_0 \mp \mathbf{n}(\beta)] + \varkappa \delta \mathbf{e}_\pm(\xi_\pm), \quad (32)$$

where

$$\mathbf{n}(\beta) = (\sin 2\beta)\mathbf{b}_1 + (\cos 2\beta)\mathbf{b}_3$$

and

$$\delta\mathbf{e}_\pm(\xi; \beta) = \left(\sin I_\pm(\xi)\right) \left[-\cos(I_\pm(\xi) \pm 2\beta)\mathbf{b}_1 \pm \sin(I_\pm(\xi) \pm 2\beta)\mathbf{b}_3\right].$$

It is clear that

$$X_0(\xi^0, \xi^1) = \varkappa \xi^0 \mathbf{b}_0 + Z_0 \quad (Z_0 = \text{const}),$$

for our gauge.

To reconstruct the spatial coordinates $X_j(\xi^0, \xi^1)$ ($j = 1, 3$) of the world-sheet through the derivatives, we must add two-dimensional vector with components Z_1 and Z_3 . Taking into account our subsequent definition of the constants Z_j , we suppose that the following conditions were fulfilled:

$$I_\pm(+\infty) = \int_{-\infty}^{\infty} \rho_\pm(\eta) d\eta = \pi n_\pm, \quad (33)$$

where n_\pm are integer numbers. These conditions will be very important for our theory. Later we shall state the topological nature of the numbers n_\pm .

Let the functions $s_{j\pm}(\omega; \beta)$ be Fourier - transformations for the components of the vectors $\delta\mathbf{e}_\pm(\xi; \beta)$:

$$s_{j\pm}(\omega; \beta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi\omega} (\delta\mathbf{e}_\pm(\xi; \beta))_j d\xi, \quad j = 1, 3.$$

The Shwarz functions ρ_\pm are subjected to the conditions (33), that is why the components $(\delta\mathbf{e}_\pm(\xi; \cdot))_j$ will be Shwarz functions. Thus the components $s_{j\pm}(\omega; \cdot)$ will be well-defined Shwarz functions too. Taking into account the restriction of the space-time invariancy group from $E(1, 2)$ to $E(2)$, the final formula for spatial string coordinates X_j will be as follows ($j = 1, 3$):

$$\begin{aligned} X_j(\xi^0, \xi^1) &= Z_j - \varkappa \mathbf{n}(\beta) \xi^1 + \\ &+ i\varkappa V.p. \int_{-\infty}^{\infty} \frac{s_{j+}(\omega; \beta)}{\omega} e^{i\xi_+\omega} d\omega - i\varkappa V.p. \int_{-\infty}^{\infty} \frac{s_{j-}(\omega; \beta)}{\omega} e^{i\xi_-\omega} d\omega. \end{aligned} \quad (34)$$

Thus we have the following one-to-one correspondence:

$$\left(X_1(\xi^0, \xi^1), X_3(\xi^0, \xi^1)\right) \longleftrightarrow \left(\rho_+(\xi_+), \rho_-(\xi_-); Z_1, Z_3, \beta, \varkappa\right). \quad (35)$$

The "external" variables $(Z_1, Z_3, \beta, \varkappa)$ correspond to the translations, rotations and scale transformations of the considered strings. The following section is devoted to hamiltonian structure of suggested theory.

3 Hamiltonian structure.

Let us write the formulae for Nöether invariants of the action (1):

$$P_\mu = \gamma \int_{-\infty}^{\infty} \partial_0 X_\mu d\xi^1, \quad M_{\mu\nu} = \gamma \int_{-\infty}^{\infty} (\partial_0 X_\mu X_\nu - \partial_0 X_\nu X_\mu) d\xi^1.$$

These expressions are formal unless the convergence of the integrals will be stated. The only quantities that are interesting in our case are spatial momentum $\mathbf{P} = \mathbf{b}_1 P_1 + \mathbf{b}_3 P_3$ and the component $M = M_{13}$; that is why we investigate the convergence of these quantities only.

Proposition 4. *The quantities P_1 , P_3 and M are finite if the conditions (33) are fulfilled.*

Indeed, formulae (32) and (34) lead to the following expressions:

$$P_1 = -\gamma \varkappa \int_{-\infty}^{\infty} [\sin(2I_+(\xi) + 2\beta) + \sin(2I_-(\xi) - 2\beta)] d\xi, \quad (36)$$

$$P_3 = -\gamma \varkappa \int_{-\infty}^{\infty} [\cos(2I_+(\xi) + 2\beta) - \cos(2I_-(\xi) - 2\beta)] d\xi, \quad (37)$$

and $M = Z_1 P_3 - Z_3 P_1 + J$, where

$$\begin{aligned} J &= \gamma \varkappa^2 F_J(\rho_+, \rho_-) \equiv \gamma \varkappa^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varepsilon(\eta_1 - \eta_2) \sin I_+(\eta_1) \sin I_-(\eta_2) \times \\ &\times [\cos I_+(\eta_1) \sin I_+(\eta_2) + \cos I_-(\eta_1) \sin I_-(\eta_2)] d\eta_1 d\eta_2. \end{aligned} \quad (38)$$

The "constraints" (33) were the sufficient conditions for the convergence of integrals in right hand sides of the formulae (36), (37) and (38). This Proposition confirms the topological nature of the conditions (33): there is no any physical theory outside of "constraint surface" (33).

Proposition 5. *The quantities P_1 , P_3 , J , ρ_\pm are constrained by the following condition:*

$$\Phi(\rho_+, \rho_-, \mathbf{P}, J) \equiv \mathbf{P}^2 - \gamma J \Omega(\rho_+, \rho_-) = 0, \quad (39)$$

where $\Omega(\rho_+, \rho_-) = F_J^{-1}(\rho_+, \rho_-) F_P(\rho_+, \rho_-)$ and

$$\begin{aligned}
F_P(\rho_+, \rho_-) &= \left[\int_{-\infty}^{\infty} [\sin(2I_+(\xi) + 2\beta) + \sin(2I_-(\xi) - 2\beta)] d\xi \right]^2 +, \\
&+ \left[\int_{-\infty}^{\infty} [\cos(2I_+(\xi) + 2\beta) - \cos(2I_-(\xi) - 2\beta)] d\xi \right]^2 = \\
&= 4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin(I_+(\xi) + I_-(\xi)) \sin(I_+(\eta) + I_-(\eta)) \times \\
&\times \cos(I_+(\xi) - I_-(\xi) - I_+(\eta) + I_-(\eta)) d\xi d\eta.
\end{aligned}$$

The proof is the exclusion of the constant \varkappa from the formulae (36), (37) and (38).

We choose the constants (P_1, P_3, J) as the external variables in our theory instead the constants (β, \varkappa) . Since we introduce three variables instead two ones, the condition (39) must be imposed as the constraint. In accordance with our supposition the constant \varkappa is non-zero finite constant; for this domain the identities $\rho_{\pm} \equiv 0$ lead to the equalities $|\mathbf{P}| = 0$, $J = 0$. We extend considered dynamical system now: the inequalities $|\mathbf{P}| \neq 0$ and (or) $J \neq 0$ will be admitted in spite of the identities $\rho_{\pm} \equiv 0$ hold. Moreover the equalities $|\mathbf{P}| = 0$, $J = 0$ will be admitted for arbitrary functions ρ_{\pm} . This extension corresponds to the addition of the boundary points $\varkappa = 0$ and $\varkappa = \infty$ to the primary domain $(0, \infty)$.

Thus the hamiltonian dynamical system in our theory is defined as follows:

- the hamiltonian variables

$$(\rho_+(\xi_+), \rho_-(\xi_-); Z_1, Z_3, P_1, P_3, J);$$

- Poisson brackets

$$\{\rho_{\pm}(\xi), \rho_{\pm}(\eta)\} = \mp \delta'(\xi - \eta), \quad (40)$$

$$\{P_i, Z_j\} = \delta_{ij} \quad (41)$$

(other possible brackets are equal to zero);

- constraints (33) and (39);

- hamiltonian

$$H = \frac{1}{2} \int_{-\infty}^{\infty} [\rho_+^2(\xi) + \rho_-^2(\xi)] d\xi + l(\xi^0)\Phi,$$

where the function l is a lagrange multiplier.

Note that dynamical variable J annihilates all Poisson brackets¹. The phase space will be as follows:

$$\mathcal{H} = \mathcal{H}_+ \times \mathcal{H}_- \times \mathcal{H}_p,$$

where \mathcal{H}_{\pm} are the phase spaces of free two-dimensional fields ρ_{\pm} and \mathcal{H}_p is the phase space of a free particle on a plane. The model is non-trivial because of the constraint (39), which entangles the internal and external variables. This constraint selects the "Regge trajectories" $J = \alpha' \mathbf{P}^2$, where the slope α' depends on the internal variables only. Topological constraints (33) select the symplectic sheets in the spaces \mathcal{H}_{\pm} .

In the next section we investigate the degrees of freedom which correspond to "internal" variables $(\rho_+(\xi_+), \rho_-(\xi_-))$.

4 Spectral problems and N - soliton strings

Let us consider the pair of spectral problems

$$\mathcal{U}'_{\pm}(\xi^0, \xi^1; \lambda) + Q_{\pm}(\xi_{\pm})\mathcal{U}_{\pm}(\xi^0, \xi^1; \lambda) = i\lambda\sigma_3\mathcal{U}_{\pm}(\xi^0, \xi^1; \lambda). \quad (42)$$

These spectral problems are similar to the ones for Non-linear Shroedinger equation, so we shall use the results of [10]. Because the functions $\rho_{\pm} = \rho_{\pm}(\xi^1 \pm \xi^0)$ are the functions from Shwarz space, we can define the matrices $\mathcal{U}^{[\pm]}(\xi^0, \xi^1; \lambda)$ by means of the asymptotical conditions

$$\mathcal{U}^{[\pm]}(\xi^0, \xi^1; \lambda) \xrightarrow{\xi^1 \rightarrow \pm\infty} \exp(i\lambda\xi^1\sigma_3). \quad (43)$$

We omit the low indices \pm , which mark the spectral problems (42), in the formula (43) as well as in all the following formulae, where such notations don't lead to any ambiguities. The monodromy martices $\mathfrak{M}(\lambda)$

$$\mathfrak{M}_{\pm}(\lambda) = \begin{pmatrix} \mathfrak{a}_{\pm}(\lambda) & \mathfrak{b}_{\pm}(\lambda) \\ -\bar{\mathfrak{b}}_{\pm}(\lambda) & \bar{\mathfrak{a}}_{\pm}(\lambda) \end{pmatrix}$$

¹This definition is motivated by the fact that the value J in non-relativistic case is the only non-zero component of Luban'sky - Pauli vector.

are defined as follows:

$$\mathcal{U}^{[+]}(\xi; \lambda) = \mathcal{U}^{[-]}(\xi; \lambda) \mathfrak{M}(\lambda). \quad (44)$$

The conditions (33) mean that

$$\mathfrak{M}_{\pm}(0) = (-1)^{n_{\pm}}. \quad (45)$$

Because the group $SO(2)$ is infinitely connected group, the numbers n_{\pm} have topological nature.

There exists a one - to - one correspondence between the coefficients ρ_{\pm} and the scattering data of the spectral problems (42) [10]. Below in this section we consider the N - soliton case only so that the equalities

$$\mathfrak{b}_{\pm}(\lambda) \equiv 0$$

take place. This supposition means that the functions ρ can be reconstructed through the discrete spectrum λ_n , c_n , where $n = n_{\pm} = 1, \dots, N_{\pm}$. The constants λ_n are the eigenvalues of the spectral problems (42); the additional constants c_n are the proportionality coefficients for the appropriate columns of the matrices $\mathcal{U}^{[+]}(\lambda)$ and $\mathcal{U}^{[-]}(\lambda)$ at $\lambda = \lambda_n$. The reality condition for the functions ρ means that the spectrum $\{\lambda_n\}$ consists of pure imaginary points as (or) a symmetrical pairs (λ_n, λ_m) so that $\lambda_n = -\bar{\lambda}_m$; there are no any other kinds of the values λ_n in the spectrum. The reduction to N - soliton case means that the functions $\mathfrak{a}_{\pm}(\lambda)$ are integer in a complex λ - plane so that the matrices $\mathcal{U}^{[\pm]}(\xi; \lambda)$ can be reconstructed explicitly by means of the matrix Riemann problem [10]. Because $\mathcal{U}_{0\pm}(\xi) = \mathcal{U}_{\pm}^{[-]}(0, \xi; 0)$, the world-sheet can be reconstructed through the scattering data of the systems (42) too. Note that conditions (45) are fulfilled here. For N -soliton case, the numbers n_{\pm} will be the numbers of pure imaginary points in the spectrum of the corresponding spectral problem.

Let us consider some examples.

- $N_+ = N_- = 0$. It is clear that $\mathcal{U}_{0\pm}(\xi) = 1_2$ in this case so that the string is a straight line and the world-sheet is a time-like plane.
- $N_+ = N_- = 1$, $\lambda_{1\pm} = ia_{\pm}$, a_{\pm} - positive real numbers. The reconstruction of the matrix elements $u_{ij} = (\mathcal{U}_0)_{ij}$ leads to the explicit formulae

$$u_{11}(\xi) = \frac{e^{-4a\xi} - c^2}{e^{-4a\xi} + c^2}, \quad u_{12}(\xi) = -\frac{2ce^{-2a\xi}}{e^{-4a\xi} + c^2}.$$

The details can be founded in the work [9] where similar calculations have been carried out in accordance with the method [11]. The string

has two moving cusps in this case (see [7], where the pictures of both the string and the world - sheet are presented for this case).

The world-lines of the cusps form the braid in the space-time $E_{1,2}$ which is demonstrated² in Fig.1. Of course, such picture of the cusp world-lines corresponds to the self-intersected world-sheet.

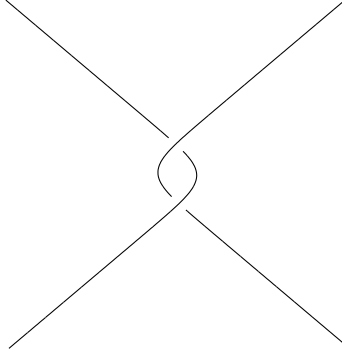


Figure 1: The braid in 2-soliton case.

- $N_+ = N_- = 2$, $\lambda_{j\pm} = ia_{j\pm}$, $a_{j\pm}$ – arbitrary positive real numbers. The example of corresponding braid is demonstrated in Fig. 2. It should be stressed that braid topology depends on the values of the constants λ_n , c_n . So, the intersection of some world-lines is possible for some conditions

$$F(\lambda_1, \dots, \lambda_N; c_1, \dots, c_N) = 0, \quad (46)$$

connected with the discrete spectrum.

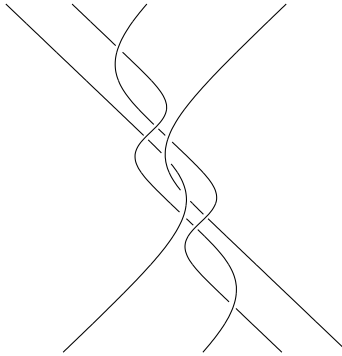


Figure 2: The example of braid in 4-soliton case.

²The axis X_0 in $E_{1,2}$ corresponds to vertical direction on the Figures 1, 2 and 4.

The inequality $\rho_+\rho_- > 0$ was fulfilled for all the examples considered above. The following Proposition is true.

Proposition 6. *If the inequality $\rho_+\rho_- > 0$ takes place, the number of cuspidal points on a string is a dynamical invariant.*

Indeed, the explicit formula for the world-sheet form **I** means that the cuspidal points correspond to the singularities of the function $\varphi(\xi^0, \xi^1)$. If the inequality $\rho_+\rho_- > 0$ takes place, the equation (7a) is conformally equivalent to the Liouville equation. The dynamics of singularity of the solutions of Liouville equation has been investigated in the work [12]. It was stated that the number of singular points is conserved by ξ^0 - dynamics and the lines of the singularities don't intersect. In spite of this fact the world-lines of the cuspidal points can intersect because the string with cuspidal points can have the intersections (see Fig.3 as some example)



Figure 3: The string with three cuspidal points: a moment before coincidence.

As it is stressed above, the suggested approach admits any sign of the product $\rho_+\rho_-$. The picture of the world-lines is more complicated when the condition $\rho_+\rho_- > 0$ was disturbed. Corresponding example for $N_+ = 2$, $N_- = 1$ is demonstrated on Fig.4

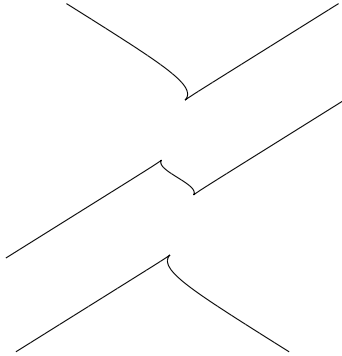


Figure 4: World-lines of the cuspidal points: example for 3-soliton case.

Thus we have the pictorial model for classical quasiparticles which can interact, be created or annihilated on a plane.

Acknowledgments. Author thank V. S. Looze for computer visualization of considered examples.

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